

# MULTILINEAR FUNCTION SERIES IN CONDITIONALLY FREE PROBABILITY WITH AMALGAMATION

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**ABSTRACT.** As in the cases of freeness and monotonic independence, the notion of conditional freeness is meaningful when complex-valued states are replaced by positive conditional expectations. In this framework, the paper presents several positivity results, a version of the central limit theorem and an analogue of the conditionally free R-transform constructed by means of multilinear function series.

## 1. Introduction

The paper addresses a topic related to conditionally free (or, shortly, using the term from [2], c-free) probability. This notion was developed in the '90's (see [1], [2]) as an extension of freeness within the framework of  $*$ -algebras endowed with not one, but two states. Namely, given a family of unital algebras  $\{\mathfrak{A}_i\}_{i \in \mathcal{I}}$ , each  $\mathfrak{A}_i$  endowed with two expectations  $\varphi_i, \psi_i : \mathfrak{A}_i \longrightarrow \mathbb{C}$ , their c-free product is the triple  $(\mathfrak{A}, \varphi, \psi)$ , where:

- (i)  $\mathfrak{A} = \ast_{i \in \mathcal{I}} \mathfrak{A}_i$  is the free product of the algebras  $\mathfrak{A}_i$ .
- (ii)  $\psi = \ast_{i \in \mathcal{I}} \psi_i$  and  $\varphi = \ast_{(\psi_i)_{i \in \mathcal{I}}} \varphi_i$  are expectations given by the relations
  - (a)  $\psi(a_1 \cdots a_n) = 0$
  - (b)  $\varphi(a_1 \cdots a_n) = \varphi_{\varepsilon(1)}(a_1) \cdots \varphi_{\varepsilon(n)}(a_n)$

for all  $a_j \in \mathfrak{A}_{\varepsilon(j)}$ ,  $j = 1, \dots, n$  such that  $\psi_{\varepsilon(j)}(a_j) = 0$  and  $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$ .

A key result is that if the  $\mathfrak{A}_i$  are  $*$ -algebras and  $\varphi_i, \psi_i$  are positive functionals, then  $\varphi$  and  $\psi$  are also positive.

In [6], the positivity of the free product maps  $\varphi, \psi$  is proved for the case when  $\varphi_1, \varphi_2$  are positive conditional expectations in a common  $C^*$ -subalgebra, but  $\psi_1, \psi_2$  remain positive  $\mathbb{C}$ -valued maps. A more general situation is indeed discussed (see Theorem 3, Section 6, from [6]), but the question if  $\varphi, \psi$  are positive for  $\varphi_{1,2}, \psi_{1,2}$  arbitrary positive conditional expectations is left unanswered.

A first answer was given in [8], where we showed that for  $\mathfrak{A}$  a  $*$ -algebra, the analogous construction with both  $\varphi$  and  $\psi$  valued in a  $C^*$ -subalgebra  $\mathfrak{B}$  of  $\mathfrak{A}$  still retains the positivity. The present paper further develops this result (see Theorem 2.3) and also demonstrates the use of multilinear function series in c-free setting.

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In [2] is constructed a  $c$ -free version of Voiculescu's  $R$ -transform, which we will call the  ${}^cR$ -transform, with the property that  ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$  if  $X$  and  $Y$  are  $c$ -free elements from the algebra  $\mathfrak{A}$  relative to  $\varphi$  and  $\psi$  (i.e. the relations (a) and (b) from the definition of the  $c$ -free product hold true for the subalgebras generated by  $X$  and  $Y$ .)

The apparatus of multilinear function series is used in recent work of K. Dykema ([3] and [4]) to construct suitable analogues for the  $R$  and  $S$ -transforms in the framework of freeness with amalgamation. We will show that this construction is also appropriate for the  ${}^cR$ -transform mentioned above. The techniques used differ from the ones of [3], the Fock space type construction being substituted by combinatorial techniques similar to [2] and [7]. Particularly, Theorems 3.3 and 3.6 contain new (shorter) proofs of the results 6.1–6.13 from [3].

The paper is structured in four sections. In Section 2 are stated the basic definitions and are proved the main positivity results. Section 3 describes the construction and the basic property of the multilinear function series  ${}^cR$ -transform and Section 4 treats the central limit theorem and a related positivity property.

## 2. Definitions and positivity results

**Definition 2.1.** Let  $\{\mathfrak{A}_i\}_{i \in \mathcal{I}}$  be a family of algebras, all containing the subalgebra  $\mathfrak{B}$ . Suppose  $\mathfrak{D}$  is a subalgebra of  $\mathfrak{B}$  and  $\Psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{D}$  and  $\Phi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$  are conditional expectations,  $i \in \mathcal{I}$ . We say that the triple  $(\mathfrak{A}, \Phi, \Psi) = \ast_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$  is the conditionally free product with amalgamation over  $(\mathfrak{B}, \mathfrak{D})$ , or shortly, the  $c$ -free product, of the triples  $(\mathfrak{A}_i, \Phi_i, \Psi_i)_{i \in \mathcal{I}}$  if

- (1)  $\mathfrak{A}$  is the free product with amalgamation over  $\mathfrak{B}$  of the family  $(\mathfrak{A}_i)_{i \in \mathcal{I}}$
- (2)  $\Psi = \ast_{i \in \mathcal{I}} \Psi_i$  and  $\Phi = \ast_{(\Psi_i)_{i \in \mathcal{I}}} \Phi_i$  are determined by the relations

$$\begin{aligned} \Psi(a_1 a_2 \dots a_n) &= 0 \\ \Phi(a_1 a_2 \dots a_n) &= \Phi(a_1) \Phi(a_2) \dots \Phi(a_n), \end{aligned}$$

for all  $a_i \in \mathfrak{A}_{\varepsilon(i)}$ ,  $\varepsilon(i) \in \mathcal{I}$  such that  $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$  and  $\Psi_{\varepsilon(i)}(a_i) = 0$ .

When  $\mathfrak{D} = \mathbb{C}$ , this definition reduces to the one given in [6]. When both  $\mathfrak{B}$  and  $\mathfrak{D}$  are equal to  $\mathbb{C}$ , this definition was given in [2].

When discussing positivity, we need a  $\ast$ -structure on our algebras. We will demand that  $\mathfrak{B}$  and  $\mathfrak{D}$  be  $C^\ast$ -algebras, while  $\mathfrak{A}_i$  and  $\mathfrak{A}$  are only required to be  $\ast$ -algebras.

The following results are slightly modified versions of Lemma 6.4 and Theorem 6.5 from [8].

**Lemma 2.2.** Let  $\mathfrak{B}$  be a  $C^\ast$ -algebra and  $\mathfrak{A}_1, \mathfrak{A}_2$  be two  $\ast$ -algebras containing  $\mathfrak{B}$  as a  $\ast$ -subalgebra, endowed with positive conditional expectations  $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$ ,  $j = 1, 2$ . If  $a_1, \dots, a_n \in \mathfrak{A}_1$ ,  $a_{n+1}, \dots, a_{n+m} \in \mathfrak{A}_2$  and  $A = (A_{i,j})_{i,j} \in M_{n+m}(\mathfrak{B})$  is the matrix with the entries

$$A_{i,j} = \begin{cases} \Phi_1(a_i^\ast a_j) & \text{if } i, j \leq n \\ \Phi_1(a_i^\ast) \Phi_2(a_j) & \text{if } i \leq n, j > n \\ \Phi_2(a_i^\ast) \Phi_1(a_j) & \text{if } i > n, j \leq n \\ \Phi_2(a_i^\ast a_j) & \text{if } i, j > n \end{cases}$$

then  $A$  is positive.

*Proof.* The vector space  $\mathfrak{E} = \mathfrak{B} \oplus \ker(\Phi_1) \oplus \ker(\Phi_2)$  has a  $\mathfrak{B}$ -bimodule structure given by the algebraic operations on  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$ . Consider the  $\mathfrak{B}$ -sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{B}$$

determined by the relations:

$$\begin{aligned} \langle b_1, b_2 \rangle &= b_1^* b_2, \text{ for } b_1, b_2 \in \mathfrak{B} \\ \langle u_j, v_j \rangle &= \Phi_j(u_j^* v_j), \text{ for } u_j, v_j \in \ker(\Phi_j), j = 1, 2 \\ \langle u_1, u_2 \rangle &= \langle u_2, u_1 \rangle = 0 \text{ for } u_1 \in \ker(\Phi_1), \text{ and } u_2 \in \ker(\Phi_2). \\ \langle b, u_j \rangle &= \langle u_j, b \rangle = 0 \text{ for all } b \in \mathfrak{B}, u_j \in \mathfrak{A}_j \end{aligned}$$

With this notation, we have that  $A_{i,j} = \langle a_i, a_j \rangle$ , hence it suffices to show that  $\langle a, a \rangle \geq 0$  for all  $a \in \mathfrak{E}$ .

Indeed, for an element  $a = b + u_1 + u_2$  with  $b \in \mathfrak{B}, u_j \in \ker(\Phi_j), j = 1, 2$ , we have:

$$\begin{aligned} \langle a, a \rangle &= \langle b + u_1 + u_2, b + u_1 + u_2 \rangle \\ &= \langle b, b \rangle + \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle \\ &= b^* b + \Phi_1(u_1^* u_1) + \Phi_2(u_2^* u_2) \\ &\geq 0 \end{aligned}$$

□

**Theorem 2.3.** *Let  $\mathfrak{B}$  be a  $C^*$ -algebra and  $\mathfrak{D}$  a  $C^*$ -subalgebra of  $\mathfrak{B}$ . Suppose that  $\mathfrak{A}_1, \mathfrak{A}_2$  are  $*$ -algebras containing  $\mathfrak{B}$ , each endowed with two positive conditional expectations  $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$ , and  $\Psi_j : \mathfrak{A}_j \longrightarrow \mathfrak{D}$ ,  $j = 1, 2$  and consider the  $c$ -free product  $(\mathfrak{A}, \Phi, \Psi) = *_{i=1,2}(\mathfrak{A}_i, \Phi_i, \Psi_i)$ .*

*Then the maps  $\Phi$  and  $\Psi$  are positive.*

*Proof.* The positivity of  $\Psi$  is by now a classical result in the theory of free probability with amalgamation over a  $C^*$ -algebra (for example, see [9], Theorem 3.5.6). For the positivity of  $\Phi$  we have to show that  $\Phi(a^* a) \geq 0$  for any  $a \in \mathfrak{A}$ .

Any element of  $\mathfrak{A}$  can be written as

$$a = \sum_{k=1}^N s_{1,k} \cdots s_{n(k),k},$$

where  $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$   $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$ .

Writing

$$s_{(j,k)} = s_{(j,k)} - \Psi(s_{(j,k)}) + \Psi(s_{(j,k)})$$

and expanding the product, we can consider  $a$  of the form

$$a = d + \sum_{k=1}^N a_{1,k} \cdots a_{n(k),k}$$

with  $d \in \mathfrak{D} \subset \mathfrak{B}$  and  $a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$  such that  $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$  and  $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$ .

Therefore

$$\begin{aligned}\Phi(a^*a) &= \Phi\left(d^*d + d^*\left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right] + \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^* d + \right. \\ &\quad \left. \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]^* \left[\sum_{k=1}^N a_{1,k} \dots a_{n(k),k}\right]\right).\end{aligned}$$

Since  $\Phi$  is a conditional expectation and  $d \in \mathfrak{D} \subset \mathfrak{B}$ , the above equality becomes

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N d^* \Phi(a_{1,k} \dots a_{n(k),k}) + \sum_{k=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^*) d \\ &\quad + \sum_{k,l=1}^N \Phi(a_{n(k),k}^* \dots a_{1,k}^* a_{1,l} \dots a_{n(l),l}).\end{aligned}$$

Using the definition of the conditionally free product with amalgamation over  $\mathfrak{B}$  and that  $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$  for all  $j, k$ , one further has

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N \Phi(d^* a_{1,k}) \Phi(a_{2,k}) \dots \Phi(a_{n(k),k}) \\ &\quad + \sum_{k=1}^N \Phi(a_{n(k),k}^* \dots \Phi(a_{2,k}^*) \Phi(a_{1,k}^*) d) \\ &\quad + \sum_{k,l=1}^N [\Phi(a_{n(k),k}^* \dots \Phi(a_{2,k}^*)] \Phi(a_{1,k}^* a_{1,l}) \Phi(a_{2,l}) \dots \Phi(a_{n(l),l})\end{aligned}$$

that is

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^N \Phi(d^* a_{1,k}) [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})] \\ &\quad + \sum_{k=1}^N [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})]^* \Phi(a_{1,k}^* d) \\ &\quad + \sum_{k,l=1}^N [\Phi(a_{2,k}) \dots \Phi(a_{n(k),k})]^* \Phi(a_{1,k}^* a_{1,l}) [\Phi(a_{2,l}) \dots \Phi(a_{n(l),l})]\end{aligned}$$

From Lemma 2.2, the matrix  $S = (\Phi(a_{1,i}^* a_{1,j}))_{i,j=1}^{N+1}$  is positive in  $M_{N+1}(\mathfrak{B})$ , therefore

$$S = T^*T, \text{ for some } T \in M_{N+1}(\mathfrak{B}).$$

Denote now  $a_{1,N+1} = d$  and  $v_k = \Phi(a_{2,k}) \dots \Phi(a_{n(k),k})$ . The identity for  $\Phi(a^*a)$  becomes:

$$\begin{aligned} \Phi(a^*a) &= (v_1, \dots, v_N, 1)^* T^* T(v_1, \dots, v_N, 1) \\ &\geq 0, \text{ as claimed.} \end{aligned}$$

□

**Theorem 2.4.** Assume that  $\mathcal{I} = \bigcup_{j \in \mathfrak{J}} \mathcal{I}_j$  is a partition of  $\mathcal{I}$ . Then:

$$*_{j \in \mathfrak{J}} (*_{i \in \mathcal{I}_j} (\mathfrak{A}_i, \Phi_i, \Psi_i)) = *_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$$

*Proof.* The proof is identical to the proofs of similar results in [6] and [2]. Consider  $a_i \in \mathfrak{A}_{\varepsilon(i)}$ ,  $1 \leq i \leq m$  such that  $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(m)$  and  $\Psi_{\varepsilon(i)}(a_i) = 0$ . Let  $1 = i_0 < i_1 < \dots < i_k = m$  and  $\mathfrak{J}_l = \{\varepsilon(i), i_{l-1} \leq i < i_l\}$ . Since

$$(*_{j \in \mathfrak{J}_l} \Psi_j)((a_{i_{l-1}} \dots a_{i_l})) = 0,$$

it suffices to show that

$$\Phi(a_1 \dots a_m) = \prod_{l=1}^k [(*_{(j), j \in \mathfrak{J}_l} \Phi_j)(a_{i_{l-1}} \dots a_{i_l})].$$

But

$$\Phi(a_1 \dots a_m) = \Phi_{\varepsilon(1)}(a_1) \dots \Phi_{\varepsilon(m)}(a_m)$$

while, since  $\Psi_{\varepsilon(i)}(a_i) = 0$ ,

$$(*_{(j), j \in \mathfrak{J}_l} \Phi_j)(a_{i_{l-1}} \dots a_{i_l}) = \Phi_{i_{l-1}}(a_{i_{l-1}}) \dots \Phi_{i_l}(a_{i_l})$$

and the conclusion follows. □

**Definition 2.5.** Let  $\mathfrak{A}$  be an algebra (respectively a  $*$ -algebra),  $\mathfrak{B}$  a subalgebra ( $*$ -subalgebra) of  $\mathfrak{A}$  and  $\mathfrak{D}$  a subalgebra ( $*$ -subalgebra) of  $\mathfrak{B}$ . Suppose  $\mathfrak{A}$  is endowed with the conditional expectations  $\Psi : \mathfrak{A} \longrightarrow \mathfrak{D}$  and  $\Phi : \mathfrak{A} \longrightarrow \mathfrak{D}$ .

- (i) The subalgebras ( $*$ -subalgebras)  $(\mathfrak{A}_i)_{i \in \mathcal{I}}$  of  $\mathfrak{A}$  are said to be *c-free* with respect to  $(\Phi, \Psi)$  if:
  - (a)  $(\mathfrak{A}_i)_{i \in \mathcal{I}}$  are free with respect to  $\Psi$ .
  - (b) if  $a_i \in \mathfrak{A}_{\varepsilon(i)}$ ,  $1 \leq i \leq m$ , are such that  $\varepsilon(1) \neq \dots \neq \varepsilon(m)$  and  $\Psi(a_i) = 0$ , then  $\Phi(a_1 \dots a_m) = \Phi(a_1) \dots \Phi(a_m)$ .
- (ii) The elements  $(X_i)_{i \in \mathcal{I}}$  of  $\mathfrak{A}$  are said to be *c-free* with respect to  $(\Phi, \Psi)$  if the subalgebras ( $*$ -subalgebras) generated by  $\mathfrak{B}$  and  $X_i$  are *c-free* with respect to  $(\Phi, \Psi)$ .

We will denote by  $\mathfrak{B}\langle \xi \rangle$  the non-commutative algebra of polynomials in the symbol  $\xi$  and with coefficients from  $\mathfrak{B}$  (the coefficients do not commute with the symbol  $\xi$ ). If  $\mathcal{I}$  is a family of indices,  $\mathfrak{B}\langle \{\xi_i\}_{i \in \mathcal{I}} \rangle$  will denote the algebra of polynomials in the non-commuting variables  $\{\xi_i\}_{i \in \mathcal{I}}$  and with coefficients from  $\mathfrak{B}$ . We will identify  $\mathfrak{B}\langle \{\xi_i\}_{i \in \mathcal{I}} \rangle$  with the free product with amalgamation over  $\mathfrak{B}$  of the family  $\{\mathfrak{B}\langle \xi_i \rangle\}_{i \in \mathcal{I}}$ .

If  $\mathfrak{A}$  is a  $*$ -algebra and  $\mathfrak{B}$  is with the  $C^*$ -algebra,  $\mathfrak{B}\langle\xi\rangle$  will also be considered with a  $*$ -algebra structure, by taking  $\xi^* = \xi$ . If  $X$  is a selfadjoint element from  $\mathfrak{A}$ , we define the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$$

given by  $\Phi_X(f(\xi)) = \Phi(f(X))$  and  $\Psi_X(f(\xi)) = \Psi(f(X))$ , for any  $f(\xi) \in \mathfrak{B}\langle\xi\rangle$ .

**Corollary 2.6.** *Suppose that  $\mathfrak{A}$  is a  $*$ -algebra and  $X$  and  $Y$  are  $c$ -free selfadjoint elements of  $\mathfrak{A}$  such that the maps  $\Phi_X, \Psi_X$  and  $\Phi_Y, \Psi_Y$  are positive. Then the maps  $\Phi_{X+Y}$  and  $\Psi_{X+Y}$  are also positive.*

*Proof.* The positivity of  $\Psi_{X+Y}$  is an immediate consequence of the fact that  $X$  and  $Y$  are free with amalgamation over  $\mathfrak{B}$  with respect to  $\Psi$ . It remains to prove the positivity of  $\Phi_{X+Y}$ .

Since the maps  $\Phi_X : \mathfrak{B}\langle\xi_1\rangle \longrightarrow \mathfrak{B}$  and  $\Phi_Y : \mathfrak{B}\langle\xi_2\rangle \longrightarrow \mathfrak{B}$  are positive, from Theorem 2.3 so is

$$\Phi_X *_{(\Psi_X, \Psi_Y)} \Phi_Y : \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle = \mathfrak{B}\langle\xi_1, \xi_2\rangle \longrightarrow \mathfrak{B}$$

Remark also that

$$i_Z : \mathfrak{B}\langle\xi\rangle \ni f(\xi) \mapsto f(X+Y) \in \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle$$

is a positive  $\mathfrak{B}$ -functional.

The conclusion follows from the fact that the  $c$ -freeness of  $X$  and  $Y$  is equivalent to

$$\Phi_{X+Y} = (\Phi_X *_{(\Psi_X, \Psi_Y)} \Phi_Y) \circ i_{X+Y}.$$

□

### 3. Multilinear function series and the ${}^cR$ -transform

Let  $\mathfrak{A}$  be a  $*$ -algebra containing the  $C^*$ -algebra  $\mathfrak{B}$ , endowed with a conditional expectation  $\Psi : \mathfrak{A} \longrightarrow \mathfrak{B}$ . If  $X$  is a selfadjoint element of  $\mathfrak{A}$ , then by the moment of order  $n$  of  $X$  we will understand the map

$$m_X^{(n)} : \underbrace{\mathfrak{B} \times \dots \times \mathfrak{B}}_{n-1 \text{ times}} \longrightarrow \mathfrak{B}$$

$$m_X^{(n)}(b_1, \dots, b_{n-1}) = \Psi(Xb_1X \dots Xb_{n-1}X)$$

If  $\mathfrak{B} = \mathbb{C}$ , then the moment-generating series of  $X$

$$m_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

encodes all the information about the moments of  $X$ . For  $\mathfrak{B} \neq \mathbb{C}$ , the straightforward generalization

$$\overline{m}_X(z) = \sum_{n=0}^{\infty} \Psi(X^n) z^n$$

generally fails to keep track of all the possible moments of  $X$ . A solution to this inconvenience was proposed in [3], namely the moment-generating multilinear function series of  $X$ . Before defining this notion, we will briefly recall the construction and several results on multilinear function series.

Let  $\mathfrak{B}$  be an algebra over a field  $K$ . We set  $\tilde{\mathfrak{B}}$  equal to  $\mathfrak{B}$  if  $\mathfrak{B}$  is unital and to the unitization of  $\mathfrak{B}$  otherwise. For  $n \geq 1$ , we denote by  $\mathcal{L}_n(\mathfrak{B})$  the set of all  $K$ -multilinear maps

$$\omega_n : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

A *formal multilinear function series* over  $\mathfrak{B}$  is a sequence  $\omega = (\omega_0, \omega_1, \dots)$ , where  $\omega_0 \in \mathfrak{B}$  and  $\omega_n \in \mathcal{L}_n(\mathfrak{B})$  for  $n \geq 1$ . According to [3], the set of all multilinear function series over  $\mathfrak{B}$  will be denoted by  $Mul[[\mathfrak{B}]]$ .

For  $\alpha, \beta \in Mul[[\mathfrak{B}]]$ , the *formal sum*  $\alpha + \beta$  and the *formal product*  $\alpha\beta$  are the elements from  $Mul[[\mathfrak{B}]]$  defined by:

$$\begin{aligned} (\alpha + \beta)_n(b_1, \dots, b_n) &= \alpha_n(b_1, \dots, b_n) + \beta_n(b_1, \dots, b_n) \\ (\alpha\beta)_n(b_1, \dots, b_n) &= \sum_{k=0}^n \alpha_k(b_1, \dots, b_k) \beta_{n-k}(b_{k+1}, \dots, b_n) \end{aligned}$$

for any  $b_1, \dots, b_n \in \mathfrak{B}$ .

If  $\beta_0 = 0$ , then the *formal composition*  $\alpha \circ \beta \in Mul[[\mathfrak{B}]]$  is defined by

$$(\alpha \circ \beta)_0 = \alpha_0$$

and, for  $n \geq 1$ , by

$$(\alpha \circ \beta)_n(b_1, \dots, b_n) = \sum_{k=1}^n \sum \alpha_k(\beta_{p_1}(b_1, \dots, b_{p_1}), \dots, \beta_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}))$$

where the second summation is done over all  $k$ -tuples  $p_1, \dots, p_k \geq 1$  such that  $p_1 + \cdots + p_k = n$  and  $q_j = p_1 + \cdots + p_{j-1}$ .

One can work with elements of  $Mul[[\mathfrak{B}]]$  as if they were formal power series. The relevant properties are described in [3], Proposition 2.3 and Proposition 2.6. As in [3], we use  $1$ , respectively  $I$ , to denote the identity elements of  $Mul[[\mathfrak{B}]]$  relative to multiplication, respectively composition. In other words,  $1 = (1, 0, 0, \dots)$  and  $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$ . We will also use the fact that an element  $\alpha \in Mul[[\mathfrak{B}]]$  has an inverse with respect to formal composition, denoted  $\alpha^{(-1)}$ , if and only if  $\alpha$  has the form  $(0, \alpha_1, \alpha_2, \dots)$  with  $\alpha_1$  an invertible element of  $\mathcal{L}_1(\mathfrak{B})$ .

**Definition 3.1.** *With the above notation, the moment-generating multilinear function series  $\mathcal{M}_X$  of  $X$  is the element of  $Mul[[\mathfrak{B}]]$  such that:*

$$\begin{aligned} \mathcal{M}_{X,0} &= \Psi(X) \\ \mathcal{M}_{X,n}(b_1, \dots, b_n) &= \Psi(Xb_1X \cdots Xb_nX). \end{aligned}$$

Given an element  $\alpha \in Mul[[\mathfrak{B}]]$ , the multilinear function series  $R_\alpha$  is defined by the following equation (see [3], Def 6.1):

$$R_\alpha = \left( (1 + \alpha I)^{-1} \right) \circ (I + I\alpha I)^{(-1)}. \quad (3.1)$$

A key property of  $R$  is that for any  $X, Y \in \mathfrak{A}$  free over  $\mathfrak{B}$ , we have

$$R_{\mathcal{M}_{X+Y}} = R_{\mathcal{M}_X} + R_{\mathcal{M}_Y}. \quad (3.2)$$

These relations were proved earlier in the particular case  $\mathfrak{B} = \mathbb{C}$ . One can also describe  $R_\alpha$  by combinatorial means, via the recurrence relation

$$\begin{aligned} \alpha_n(b_1, \dots, b_n) &= \sum_{k=0}^n \sum R_{\alpha, k} \left( [b_1 \alpha_{p(1)}(b_3, \dots, b_{i_1-2}) b_{i_1-1}], \dots \right. \\ &\quad \left. \dots, [b_{i(k-1)} \alpha_{p(k)}(b_{i(k-1)+1}, \dots, b_{i(k)-2}) b_{i(k)-1}] \right) b_{i(k)} \alpha_{n-i_k}(b_{i_{k+1}}, \dots, b_n) \end{aligned}$$

where the second summation is done over all  $1 = i(0) < i(1) < \dots < i(k) \leq n$  and  $p(k) = i(k) - i(k-1) - 2$ .

Following an idea from [2], the above equation can be graphically illustrated by the picture:

$$\boxed{\phantom{000000}} = \sum \boxed{\phantom{00}} \boxed{\phantom{00}} \dots \boxed{\phantom{00}} \boxed{\phantom{000000}}$$

In the case of scalar  $c$ -free probability, an analogue of the Voiculescu's  $R$ -transform is developed in [2]. In order to avoid confusions, we will denote it by  ${}^cR$ .

The  ${}^cR$ -transform has the property that it linearizes the  $c$ -free convolution of pairs of compactly supported measures. In particular, if  $X$  and  $Y$  are  $c$ -free elements from some algebra  $\mathfrak{A}$ , then

$${}^cR_{X+Y} = {}^cR_X + {}^cR_Y.$$

If the  $*$ -algebra  $\mathfrak{A}$  is endowed with the  $\mathbb{C}$ -valued states  $\varphi, \psi$  and  $X$  is a selfadjoint element of  $\mathfrak{A}$ , then (see [2]), the coefficients  $\{{}^cR_m\}_m \geq 0$  of  ${}^cR_X$  are defined by the recurrence:

$$\varphi(X^n) = \sum_{k=1}^n \sum_{\substack{l(1), \dots, l(k) \geq 0 \\ l(1) + \dots + l(k) = n-k}} {}^cR_k \cdot \psi(X^{l(1)}) \dots \psi(X^{l(k-1)}) \varphi(X^{l(k)})$$

equation that can be graphically illustrated by the picture, where the dark boxes stand for the application of  $\varphi$  and the light ones for the application of  $\psi$ :

$$\blacksquare = \sum \boxed{\phantom{00}} \boxed{\phantom{00}} \dots \boxed{\phantom{00}} \blacksquare$$

The above considerations lead to the following definition:

**Definition 3.2.** Let  $\beta, \gamma \in \text{Mul}[[\mathfrak{B}]]$ . The multilinear function series  ${}^cR_{\beta, \gamma}$  is the element of  $\text{Mul}[[\mathfrak{B}]]$  defined by the recurrence relation

$$\begin{aligned} \beta_n(b_1, \dots, b_n) &= \sum_{k=0}^n \sum {}^cR_{\beta, \gamma, k} \left( [b_1 \gamma_{p(1)}(b_3, \dots, b_{i_1-2}) b_{i_1-1}], \dots \right. \\ &\quad \left. \dots, [b_{i(k-1)} \gamma_{p(k)}(b_{i(k-1)+1}, \dots, b_{i(k)-2}) b_{i(k)-1}] \right) b_{i(k)} \beta_{n-i_k}(b_{i_{k+1}}, \dots, b_n) \end{aligned}$$

where the second summation is done over all  $1 = i(0) < i(1) < \dots < i(k) \leq n$  and  $p(k) = i(k) - i(k-1) - 2$ .



The following analytical description of  ${}^cR_{\beta,\gamma}$  also shows that it is unique and well-defined:

**Theorem 3.3.** *For any  $\beta, \gamma \in \text{Mul}[[\mathfrak{B}]]$ ,*

$${}^cR_{\beta,\gamma} = [\beta(1 + I\beta)^{-1}] \circ (I + I\gamma I)^{(-1)} \quad (3.3)$$

Before proving the theorem, remark that the right-hand side of (3.3) is well-defined and unique, since  $1 + I\gamma$  is invertible with respect to the formal multiplication,  $I + I\beta I$  is invertible with respect to formal composition and its inverse has 0 as first component (see [3]). We will need the following auxiliary result:

**Lemma 3.4.** *Let  $\beta$  be an element of  $\text{Mul}[[\mathfrak{B}]]$  and  $I$  the identity element with respect to formal composition,  $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$ .*

(i) *the multilinear function series  $I\beta$  is given by:*

$$\begin{aligned} (I\beta)_0 &= 0 \\ (I\beta)_n(b_1, \dots, b_n) &= b_1\beta_{n-1}(b_2, \dots, b_n) \end{aligned}$$

(ii) *the multilinear function series  $I\beta I$  is given by*

$$\begin{aligned} (I\beta I)_0 &= 0 \\ (I\beta I)_1(b_1) &= 0 \\ (I\beta I)_n(b_1, \dots, b_n) &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

*Proof.* Since  $I = (0, id_{\mathfrak{B}}, 0, \dots)$ , one has:

$$(I\beta)_0 = I_0\beta_0 = 0.$$

If  $n \geq 1$ ,

$$\begin{aligned} (I\beta)_n(b_1, \dots, b_n) &= \sum_{k=0}^n I_k(b_1, \dots, b_k)\beta_{n-k}(b_{k+1}, \dots, b_n) \\ &= I_1(b_1)\beta_{n-1}(b_2, \dots, b_n) \\ &= b_1\beta_{n-1}(b_2, \dots, b_n). \end{aligned}$$

For  $I\beta I$ , the same computations give:

$$\begin{aligned} (I\beta I)_0 &= (I\beta)_0 I_0 = 0 \\ (I\beta I)_1 &= (I\beta)_0 I_1(b_1) + (I\beta)_1(b_1) I_0 \\ &= 0. \end{aligned}$$

If  $n \geq 2$ , one has:

$$\begin{aligned} (I\beta I)_n(b_1, \dots, b_n) &= \sum_{k=0}^n (I\beta)_k(b_1, \dots, b_k) I_{n-k}(b_{k+1}, \dots, b_n) \\ &= (I\beta)_{n-1}(b_1, \dots, b_{n-1}) I_1(b_n) \\ &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

□

*Proof of the Theorem 3.3:* Set  $\sigma = I + I\beta I$ . Then

$$({}^cR_{\beta,\gamma} \circ \sigma)_n(b_1, \dots, b_n) = \sum_{k=1}^n \sum_{\substack{p_1, \dots, p_k \geq 1 \\ p_1 + \dots + p_k = n}} {}^cR_{\beta,\gamma,k} \left( \sigma_{p_1}(b_1, \dots, b_{p_1}), \dots, \right. \\ \left. \sigma_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}) \right)$$

where  $q_i = p_1 + \dots + p_{i-1}$ .

From Lemma (3.4)(ii), we have that

$$\sigma_n(b_1, \dots, b_n) = (I + I\beta I)_n(b_1, \dots, b_n)$$

therefore Definition 3.2 is equivalent to

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) b_{k+1} \beta_{n-k-2}(b_{k+2}, \dots, b_n)$$

Considering now Lemma 3.4(i), the above relation becomes

$$\beta_n(b_1, \dots, b_n) = \sum_{k=0}^n ({}^cR_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \dots, b_k)) (I + I\beta)_{n-k}(b_{k+1}, \dots, b_n)$$

therefore

$$\beta = [{}^cR_{\beta,\gamma} \circ (I + I\gamma I)] (1 + I\beta)$$

which is equivalent to (3.3).  $\square$

*Remark 3.5.* Up to a shift in the coefficients, equation (3.3) is similar to the result in the case  $\mathfrak{B} = \mathbb{C}$  from [2], Theorem 5.1.

Let  $X$  be a selfadjoint element of  $\mathfrak{A}$ . If  $\mathfrak{A}$  is endowed with two  $\mathfrak{B}$ -valued conditional expectations  $\Phi, \Psi$ , the element  $X$  will have two moment-generating multilinear function series, one with respect to  $\Psi$ , that we will denote by  $\mathcal{M}_X$ , and one with respect to  $\Phi$ , denoted  $\mathfrak{M}_X$ . For brevity, we will use the notation  ${}^cR_X$  for the multilinear function series  ${}^cR_{\mathcal{M}_X, \mathfrak{M}_X}$ .

**Theorem 3.6.** *Let  $X$  and  $Y$  be two elements of  $\mathfrak{A}$  that are  $c$ -free with respect to the pair of conditional expectations  $(\Phi, \Psi)$ . Then*

$${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$$

*Proof.* Let  $\mathcal{A}$  be an algebra containing  $\mathfrak{B}$  as a subalgebra and endowed with the conditional expectations  $\Phi, \Psi : \mathcal{A} \rightarrow \mathfrak{B}$ . Consider the set  $\mathcal{A}_0 = \mathcal{A} \setminus \mathfrak{B}$  (set difference). For  $n \geq 1$  define the maps

$${}^c r : \underbrace{\mathcal{A}_0 \times \dots \times \mathcal{A}_0}_{n \text{ times}} \rightarrow \mathfrak{B}$$

given by the recurrence formula:

$$\Phi(a_1 \dots a_n) = \sum_{k=1}^n \sum_{\substack{l(1) < \dots < l(k) \\ 1 < l(1), l(k) \leq n}} {}^c r_k \left( a_1 [\Psi(a_2 \dots a_{l(1)-1})], \dots, \right. \\ \left. \dots, a_{l(k-1)} [\Psi(a_{l(k-1)+1} \dots a_{l(k)_1})], a_{l(k)} [\Phi(a_{l(k)+1} \dots a_n)] \right)$$

Note that  ${}^c r_n$  is well defined, and that, for any  $b_1, \dots, b_n \in \mathfrak{B}$ ,

$${}^c r_{n+1}(X, b_1 X, \dots, b_n X) = {}^c R_{X,n}(b_1, \dots, b_n). \quad (3.4)$$

As in Section 2, consider  $\mathfrak{B}\langle \xi_i \rangle$ , the noncommutative algebras of polynomials in the symbols  $\xi_i, i = 1, 2$  and with coefficients from  $\mathfrak{B}$  and the conditional expectations

$$\Phi_X, \Psi_X : \mathfrak{B}\langle \xi_1 \rangle \longrightarrow \mathfrak{B}$$

given by

$$\begin{aligned} \Phi_X(f(\xi_1)) &= \Phi(f(X)) \\ \Psi_X(f(\xi_1)) &= \Psi(f(X)) \end{aligned}$$

and their analogues  $\Phi_Y, \Psi_Y$  for  $\mathfrak{B}\langle \xi_2 \rangle$ .

On  $\mathfrak{B}\langle \xi_1, \xi_2 \rangle$ , identified to  $\mathfrak{B}\langle \xi_1 \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2 \rangle$ , consider the conditional expectations  $\Psi_0, \Phi_0, \varphi$  given by:

$$\begin{aligned} \Psi_0 &= \Psi_X * \Psi_Y \\ \Phi_0(f(\xi_1, \xi_2)) &= \Phi(f(X, Y)) \\ \varphi(a_1 a_2 \dots a_n) &= \sum_{k=1}^n \sum_{\substack{l(1) < \dots < l(k) \\ 1 < l(1), l(k) \leq n}} \rho_k \left( a_1 [\Psi_0(a_2 \dots a_{l(1)-1})], \dots, \right. \\ &\quad \left. \dots, a_{l(k-1)} [\Psi_0(a_{l(k-1)+1} \dots a_{l(k)_1})], a_{l(k)} [\varphi(a_{l(k)+1} \dots a_n)] \right) \end{aligned}$$

where  $a_1, \dots, a_n$  are elements of the set  $\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 = \mathfrak{B}\langle \xi_1 \rangle \cup \mathfrak{B}\langle \xi_2 \rangle \setminus \mathfrak{B}$ , and the maps

$$\rho_n : \underbrace{\mathfrak{B}\langle \xi_1, \xi_2 \rangle_0 \times \dots \times \mathfrak{B}\langle \xi_1, \xi_2 \rangle_0}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

are given by:

$$\rho_n(a_1, \dots, a_n) = \begin{cases} {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_1 \rangle \\ {}^c r(a_1, \dots, a_n) & \text{if all } a_1, \dots, a_n \in \mathfrak{B}\langle \xi_2 \rangle \\ 0 & \text{otherwise} \end{cases}.$$

We will show that  $\varphi = \Phi_0$ , in particular  $\varphi$  is also well-defined. Consider the element  $a \in \mathfrak{B}\langle \xi_1, \xi_2 \rangle$  of the form  $a = a_1 \dots a_n$  with  $a_j \in \mathfrak{B}\langle \xi_{\varepsilon(j)} \rangle$ , such that  $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$  and  $\Psi_0(a_j) = 0$ . The computation of  $\varphi(a_1 \dots a_n)$  is done via the recurrence relation above. Because of the definition of  $\rho$  and the fact that  $\Psi_0 = \Psi_X * \Psi_Y$ , only the term with  $k = 1$  contribute at the sum, i.e.

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \varphi(a_1 \varphi(a_2 \dots a_n)) \\ &= \varphi_{\varepsilon(1)}(a_1 \varphi(a_2 \dots a_n)) \\ &= \varphi_{\varepsilon(1)}(a_1) \varphi(a_2 \dots a_n) \end{aligned}$$

and the identity between  $\varphi$  and  $\Phi_0$  follows by induction over  $n$ .

Since  $\varphi = \Phi_0$ , the maps  $\rho_n$  and  ${}^c r_n$  are satisfying the same recurrence relation, hence

$$\rho_n(a_1, \dots, a_n) = {}^c r(a_1, \dots, a_n).$$

In particular

$$\begin{aligned}
{}^cR_{X+Y,n}(b_1, \dots, b_n) &= {}^c r_{n+1}((X+Y)b_1(X+Y) \dots (X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X+Y)b_1(X+Y) \dots (X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X)b_1(X) \dots (X)b_n(X)) + \rho_{n+1}((Y)b_1(Y) \dots (Y)b_n(Y)) \\
&= {}^cR_{X,n}(b_1, \dots, b_n) + {}^cR_{Y,n}(b_1, \dots, b_n).
\end{aligned}$$

□

#### 4. Central limit theorem

Consider the ordered set  $\langle n \rangle = \{1, 2, \dots, n\}$  and  $\pi$  a partition of  $\langle n \rangle$  with blocks  $B_1, \dots, B_m$ :

$$\langle n \rangle = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m.$$

The blocks  $B_p$  and  $B_q$  of  $\pi$  are said to be *crossing* if there exist  $i < j < k < l$  in  $\langle n \rangle$  such that  $i, k \in B_p$  and  $j, l \in B_q$ .

The partition  $\pi$  is said to be *non-crossing* if all pairs of distinct blocks of  $\pi$  are not crossing. We will denote by  $NC_2(n)$  the set of all non-crossing partitions of  $\langle n \rangle$  whose blocks contain exactly 2 elements and by  $NC_{\leq s}(n)$  the set of all non-crossing partitions of  $\langle n \rangle$  whose blocks contain at most  $s$  elements.

Let now  $\gamma$  be a non-crossing partition of  $\langle n \rangle$  and  $B$  and  $C$  be two blocks of  $\pi$ . We say that  $B$  is interior to  $C$  if there exist two indices  $i < j$  in  $\langle n \rangle$  such that  $i, j \in C$  and  $B \subset \{i+1, \dots, j-1\}$ . The block  $B$  is said to be *outer* if it is not interior to any other block of  $\gamma$ . In a non-crossing partition of  $\langle n \rangle$ , the block containing 1 is always outer.

Consider now an element  $X$  of  $\mathfrak{A}$ . Let  $\pi$  be a partition from  $NC_2(n+1)$  ( $n = \text{odd}$ ) and  $B_1 = (1, k)$  be the block of  $\pi$  containing 1. We define, by recurrence, the following expressions:

$$\begin{aligned}
V_\pi(X, b_1, \dots, b_n) &= \Psi(Xb_1V_{\pi|\{2, \dots, j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k \\
&\quad V_{\pi|\{k+1, \dots, n+1\}}(X, b_{k+1}, \dots, b_n) \\
W_\pi(X, b_1, \dots, b_n) &= \Phi(Xb_1V_{\pi|\{2, \dots, j-1\}}(X, b_2, \dots, b_{k-2})b_{k-1}X)b_k \\
&\quad W_{\pi|\{k+1, \dots, n+1\}}(X, b_{k+1}, \dots, b_n)
\end{aligned}$$

**Theorem 4.1.** (*Central Limit Theorem*) Let  $(X_n)_{n \geq 1}$  be a sequence of  $c$ -free elements of  $\mathfrak{A}$  such that:

- (1) all  $X_n$  have the same moment-generating multilinear function series,  $\mathfrak{M}$  with respect to  $\Phi$  and  $M$  with respect to  $\Psi$ .
- (2)  $\Psi(X_n) = \Phi(X_n) = 0$ .

Set

$$S_N = \frac{X_1 + \dots + X_N}{\sqrt{N}},$$

Then:

- (i)  $\lim_{N \rightarrow \infty} {}^cR_{S_N} = (0, \mathfrak{M}_1(\cdot), 0, \dots)$

- (ii)  $\lim_{N \rightarrow \infty} R_{S_N} = (0, M_1(\cdot), 0, \dots)$
- (iii) *there exist two conditional expectations  $\nu : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$ , depending only on  $M_1(\cdot)$ , and  $\mu : \mathfrak{B}\langle \xi \rangle \longrightarrow \mathfrak{B}$ , depending only on  $M_1(\cdot)$  and  $\mathfrak{M}_1(\cdot)$ , such that*

$$\begin{aligned} \lim_{N \rightarrow \infty} \Psi_{S_N} &= \nu \\ \lim_{N \rightarrow \infty} \Phi_{S_N} &= \mu \end{aligned}$$

in the weak sense; in particular,

$$\begin{aligned} \nu(\xi b_1 \xi \dots b_n \xi) &= \sum_{\pi \in NC_2(n)} V_\pi(X_1, b_1, \dots, b_n) \\ \mu(\xi b_1 \xi \dots b_n \xi) &= \sum_{\pi \in NC_2(n)} W_\pi(X_1, b_1, \dots, b_n). \end{aligned}$$

*Proof.* Let  $X$  be an element of  $\mathfrak{A}$  with the same moment generating series as  $X_j$ ,  $j \geq 1$ . As shown in [3],

$$R_{S_N} = \sum_{k=1}^N R_{\frac{X_k}{\sqrt{N}}} = N R_{\frac{X}{\sqrt{N}}}.$$

Also, from Theorem 2.4 and Theorem 3.6, it follows that

$${}^c R_{S_N} = \sum_{k=1}^N {}^c R_{\frac{X_k}{\sqrt{N}}} = N {}^c R_{\frac{X}{\sqrt{N}}}.$$

Since  $R$  and  ${}^c R$  are multilinear and  $M_0 = \mathfrak{M}_0 = 0$ , we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} {}^c R_{S_N, n} &= \lim_{N \rightarrow \infty} \frac{N}{N^{\frac{n+1}{2}}} {}^c R_{X, n} \\ &= \begin{cases} 0 & \text{if } n \neq 1 \\ \mathfrak{M}_1(\cdot) & \text{if } n = 1 \end{cases} \end{aligned}$$

and the similar relations for  $R_{S_N, n}$ , hence (i) and (ii) are proved.

For (iii) it suffices to check the relations for  $\nu(\xi b_1 \xi \dots b_n \xi)$  and  $\mu(\xi b_1 \xi \dots b_n \xi)$ , which are a trivial corollary of (i), (ii), and the recurrence formulas that define  $R$  and  ${}^c R$ .  $\square$

*Remark 4.2.* For  $\mathfrak{B} = \mathbb{C}$ , the theorem is a weaker version of Theorem 4.3 from [2]. If  $\Psi$  is  $\mathbb{C}$ -valued, then the result is similar to Corollary 5.1 from [6]. Also, under the assumptions that for some  $a, b \in \mathfrak{B}$  we have that:

$$\begin{aligned} \lim_{N \rightarrow \infty} N \Psi(X_1 \cdots X_N) &= a \\ \lim_{N \rightarrow \infty} N \Psi(X_1 \cdots X_N) &= b \end{aligned}$$

the same techniques lead to a Poisson-type limit Theorem, similar to Corollary 2, Section 5 of [6].

In the following remaining pages we will describe the positivity of the limit functionals  $\mu$  and  $\nu$  in terms of  $\Phi$  and  $\Psi$ . The central result is Corollary 4.4.

For simplicity, suppose that  $\mathfrak{B}$  is a unital  $*$ -algebra (otherwise, we can replace  $\mathfrak{B}$  by its unitisation). Consider the symbol  $\xi$ , the  $*$ -algebra  $\mathfrak{B}\langle\xi\rangle$  of polynomials in  $\xi$  with coefficients from  $\mathfrak{B}$ , as defined before, and consider also the linear space  $\mathfrak{B}\xi\mathfrak{B}$  generated by the set  $\{b_1\xi b_2; b_1, b_2 \in \mathfrak{B}\}$  with the  $\mathfrak{B}$ -bimodule structure given by

$$a_1 b_1 \xi b_2 a_2 = (a_1 b_1) \xi (b_2 a_2)$$

for all  $a_1, a_2, b_1, b_2 \in \mathfrak{B}$ .

**Lemma 4.3.** *For any positive  $\mathfrak{B}$ -sesquilinear pairing  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{B}\xi\mathfrak{B}$  there exists a positive conditional expectation  $\varphi : \mathfrak{B}\langle\xi\rangle \rightarrow \mathfrak{B}$  such that for any  $b_1, b_2 \in \mathfrak{B}$  one has that*

$$\varphi(\xi b_1^* b_2 \xi) = \langle b_1 \xi, b_2 \xi \rangle$$

*Proof.* Without loss of generality, we can suppose that  $\mathfrak{B}$  is unital (otherwise we can replace  $\mathfrak{B}$  by its unitization).

Consider the Full Fock bimodule over  $\mathfrak{B}\xi\mathfrak{B}$

$$\mathcal{F}\langle\xi\rangle = \mathfrak{B} \oplus \left( \bigoplus_{n \geq 1} \underbrace{\mathfrak{B}\xi\mathfrak{B} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{B}\xi\mathfrak{B}}_{n \text{ times}} \right)$$

with the pairing given by

$$\begin{aligned} \langle a, b \rangle &= a^* b \\ \langle a_1 \xi \otimes \cdots \otimes a_n \xi, b_1 \xi \otimes \cdots \otimes b_m \xi \rangle &= \delta_{m,n} \langle a_n \xi, \langle \dots, \langle a_1 \xi, b_1 \xi \rangle b_2 \xi \rangle, \dots, b_n \xi \rangle. \end{aligned}$$

$(a, a_j, b, b_j \in \mathfrak{B}, j = 1, \dots, n)$

Note that the  $\mathfrak{B}$ -linear operators  $A_1, A_2 : \mathcal{F}\langle\xi\rangle \rightarrow \mathcal{F}\langle\xi\rangle$  described by the relations

$$\begin{aligned} A_1 b &= \xi b \\ A_1(a_1 \xi \otimes \cdots \otimes a_n \xi b) &= \xi \otimes a_1 \xi \otimes \cdots \otimes a_n \xi b \\ A_2 b &= 0 \\ A_2(a_1 \xi \otimes \cdots \otimes a_n \xi b) &= \langle \xi, a_1 \xi \rangle a_2 \xi \otimes \cdots \otimes a_n \xi b \end{aligned}$$

are self-adjoint to each other, in the sense that

$$\langle A_1 \tilde{\zeta}_1, \tilde{\zeta}_2 \rangle = \langle \tilde{\zeta}_1, A_2 \tilde{\zeta}_2 \rangle$$

for any  $\tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathcal{F}\langle\xi\rangle$ , therefore  $S = A_1 + A_2$  is selfadjoint.

Moreover, for any  $a, b \in \mathfrak{B}$ ,

$$\begin{aligned} \langle 1, S a^* b S 1 \rangle &= \langle a S 1, b S 1 \rangle \\ &= \langle a(A_1 + A_2)1, b(A_1 + A_2)1 \rangle \\ &= \langle a \xi, b \xi \rangle \end{aligned}$$

and the conclusion follows by setting  $\varphi(p(\xi)) = \langle 1, p(S)1 \rangle$  for all  $p \in \mathfrak{B}\langle\xi\rangle$ .  $\square$

**Corollary 4.4.** *The maps  $\mu$  and  $\nu$  from Theorem 4.1 are positive if and only if for any  $b \in \mathfrak{B}$  one has that  $\Phi(X b^* b X) \geq 0$  and  $\Psi(X b^* b X) \geq 0$ .*

*Proof.* One implication is trivial, since, if  $\nu$  and  $\mu$  are positive, then

$$\Psi(Xb^*bX) = \nu(Xb^*bX) = \nu((bX)^*bX) \geq 0$$

and

$$\Phi(Xb^*bX) = \mu(Xb^*bX) = \mu((bX)^*bX) \geq 0.$$

Suppose now that  $\Phi(Xb^*bX) \geq 0$  and  $\Psi(Xb^*bX) \geq 0$  for all  $b \in \mathfrak{B}$ . We will use the same argument as in [9] and [8].

Consider the set of selfadjoint symbols  $\{\xi_i\}_{i \geq 1}$ . On each  $\mathfrak{B}$ -bimodule  $\mathfrak{B}\xi_i\mathfrak{B}$  we have the positive  $\mathfrak{B}$ -sesquilinear pairings  $\langle \cdot, \cdot \rangle_\Phi$  and  $\langle \cdot, \cdot \rangle_\Psi$  determined by

$$\begin{aligned} \langle a\xi_i, b\xi_i \rangle_\Phi &= \Phi(Xa^*bX) \\ \langle a\xi_i, b\xi_i \rangle_\Psi &= \Psi(Xa^*bX). \end{aligned}$$

As shown in Lemma 4.3, the above  $\mathfrak{B}$ -sesquilinear pairings determine positive conditional expectations  $\varphi_1, \psi_i : \mathfrak{A}_i \rightarrow \mathfrak{B}$ , where  $\mathfrak{A}_i = \mathfrak{B}\langle \xi_i \rangle$  be the  $*$ -algebras of polynomials in  $\xi$  with coefficients from  $\mathfrak{B}$ ,  $i \geq 1$ .

For  $\tau : \mathfrak{B}\langle \xi \rangle \rightarrow \mathfrak{B}$  a conditional expectation, and  $\lambda \geq 0$ , note with  $D_\lambda \tau$  the dilation with  $\lambda$  of  $\tau$ , i.e.

$$D_\lambda \tau(\xi b_1 \xi \cdots b_n \xi) = \lambda^{n+1} \tau(\xi b_1 \xi \cdots b_n \xi)$$

Remark that if  $\tau$  is positive, then  $D_\lambda \tau$  is also positive.

With the notations above, consider, as in Definition 2.1, the conditionally free product  $(\mathfrak{A}, \Phi, \Psi) = \ast_{i \in \mathcal{I}} (\mathfrak{A}_i, \Phi_i, \Psi_i)$ . The elements  $\{\xi_i\}_{i \geq 1}$  are conditionally free in  $\mathfrak{A}$ , so Theorem 4.1 implies that:

$$\begin{aligned} \mu &= \lim_{N \rightarrow \infty} \Phi_{\frac{\xi_1 + \cdots + \xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}} \Phi_{\xi_1 + \cdots + \xi_N} \\ \nu &= \lim_{N \rightarrow \infty} \Psi_{\frac{\xi_1 + \cdots + \xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}} \Psi_{\xi_1 + \cdots + \xi_N} \\ &= D_{\frac{1}{\sqrt{N}}} \left( \ast_{i=1}^N \Psi_{\xi_i} \right). \end{aligned}$$

We have that  $\ast_{i=1}^N \Psi_{\xi_i} \geq 0$  since it is the free product of states (see, for example [9]), hence the positivity of  $\nu$ .

Also, Theorem 2.4 and Corollary 2.6 imply that  $\Phi_{\xi_1 + \cdots + \xi_N} \geq 0$ , therefore  $\mu \geq 0$ .  $\square$

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### References

1. Bożejko, M. and Speicher, R.:  $\psi$ -independent and symmetrized white noises., in: *Quantum Probability and Related Topics*, **VI** (1991), 219-236, World Scientific, Singapore.
2. Bożejko, M., Leinert, M. and Speicher, R.: Convolution and Limit Theorems for Conditionally free Random Variables., in: *Pac. J. Math.* **175** (1996), 357-388.
3. Dykema, K.: Multilinear function series and transforms in Free Probability theory, in: *Advances in Mathematics* **208** (2007) 351407.
4. Dykema, K.: On the S-transform over a Banach algebra, in: *Journal of Functional Analysis* **231** (2006) 90110
5. Lance, E. C.: *Hilbert  $C^*$ -modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series **210**, Cambridge University Press 1990.
6. Młotkowski, W.: Operator-valued version of conditionally free product, in: *Studia Mathematica* **153** (1) (2002)
7. Nica, A. and Speicher, R.: *Lectures on the Combinatorics of the Free Probability*, London mathematical Society Lecture Note Series 335, Cambridge University Press 2006
8. Popa, M.: A combinatorial approach to monotonic independence over a  $C^*$ -algebra Preprint, arXiv: math.OA/0612570, 01/2007
9. Speicher, R.: Combinatorial Theory of the Free Product with amalgamation and Operator-Valued Free Probability Theory, in: *Mem. AMS*, Vol **132**, No 627 (1998)

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